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LATTICE IMPLICATIVE ALGEBRAS

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ABSTRACT

In this research paper, firstly, I introduce the concept of implicative algebras and obtain certain properties. Further, I prove that implicative algebra is equipped with a structure of a bounded lattice and prove that it is lattice implication algebra. It also observes that "→" can never be associative. Secondly, we introduce two more binary operations "+" and "−"on implicative algebra and obtain certain properties with these operations. Further, we prove that any implicative algebra is a metric space. Also, we prove that every implicative algebra can be made into the regular authomethrized algebra of Swamy (1964).

KEYWORDS: *Implicative Algebra, Concept, Prove, Regular Authomethrized Algebra*

INTRODUCTION

The concept of implicative algebra is introduced by Abbott. J.C [1] in the case of classical propositional logic, which he discusses algebraic systems with a binary operation modeled on the BOOLEAN OPERATION $x \to y = x' v\, y$, where x' is the Boolean complement of x. Such algebra is satisfying basic equalities motivated from laws of implication in the absolute propositional calculus and is called implicative algebras. Later the authors G.M. Hardegree in 1981 define implicative algebra in the case of non-classical propositional logic. The concept of lattice implication algebras is due to Xu[7]. In his paper, he introduced the concept of lattice implication algebra andquasi-lattice implication algebra as a bounded lattice satisfying a system of some axioms and studied certain properties. Later many authors like Jun et al. [2] have studied the properties of filters and fuzzy filters of lattice implication algebras and quasi-lattice implication algebras. Also Zhu.Y and Tu. W [8] have introduced an equivalent definition for lattice implication algebra in Xu. Pan.Y [6] and Yang. Xu et al. [9] also define a lattice implication algebra by combining lattice and implication algebra and also he gives some examples of lattice implication algebras.

Definition 1.2.1

Suppose R is a relation defined on a non-empty set S satisfying the following three properties.

$[O1]$ For any $a \in S$, we have aRa…………………. (Reflexive)

$[O2]$ If aRb and bRa, then $a=b$……………… (Anti-symmetric)

$[O3]$ If aRb and bRc, then aRc…………………..… (Transitive)

Then the relation R is called a **partial ordering** of S or, simply it is a partial order. Now a set S together with the partial order R, that is (S,R) is called a **partially ordered set** or, simply, an ordered set or poset. When we want to specify the relation R, the most familiar ordering relation or which is called a natural ordering relation is denoted by "$\leq$" (read "less than or equal").

Definition 1.2.2

A poset $(S, \leq)$ is called a **chain or totally (linearly) ordered set** if for all a, $b \in S$, either $a \leq b$ or $b \leq a$. On the other hand, $(S, \leq)$ is an **anti-chain** provided any two distinct elements a, $b \in S$ are incomparable, that is, neither $a \leq b$ nor $b \leq a$ in symbol $a \backslash\backslash b$.

Examples 1.2.3

- Let S be any collection of sets. The relation $\subseteq$ of set inclusion is a partial ordering of S. Specifically, $A \subseteq A$ for any set A; if $A \subseteq B$ and $B \subseteq A$ then A=B and if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.
- Consider the set N of positive integers. We say "a divides b" written $a \mid b$, if there exists an integer c such that $ac=b$. For example, 2 | 4,3 |12,7 | 21 and so on. This relation of divisibility is a partial ordering of N.
- The relation "|" of divisibility is not an ordering of the set Z of integers. Specifically, the relation is non anti-symmetric. For instance, 2|−2 and −2 | 2, but $2 \neq -2$.

Definition 1.2.4

There are two ways to define a lattice L. One way is to define L in terms of a partially ordered set. That is, a lattice L may be defined as a partially ordered set in which both $inf\ (a, b)$ and $sup(a,b)$ exists for any pair of elements $a,b \in L$. Another way is to define a lattice L axiomatically. That is, a non-empty set L closed under two binary operations called meet and join denoted by $\wedge$ and $\vee$ respectively, then $(L; \wedge, \vee)$ is called **lattice** if the following axioms hold where a,b,c are elements in L.

[L1] Commutative law

(1a) $a \wedge b = b \wedge a$

(1b) $a \vee b = b \vee a$

[L2] Associative law

(2a) $(a \wedge b) \wedge c = a \wedge (b \wedge c) \wedge (b \wedge c)$

(2b) $(a \vee b) \vee c = a \vee (b \vee c)$

[L3] Absorption law

(3a) $a \wedge (a \vee b) = a$

(3b) $a \vee (a \vee b) = a$

[L4] Idempotent law

(4a) $a \wedge a = a$

(4b) $a \vee a = a$

Definition 1.2.15

A bounded lattice (L; ∨,∧,0,1) with an order reversing involution " ' " and a binary operation "→" is called a **lattice implication algebra** if for any $x,y,z \in L$ it satisfies the following axioms.

(I1) $x \to (y \to z) = y \to (x \to z)$

(I2) $x \to x = 1$

(I3) $x \to y = y' \to x'$

(I4) $x \to y = y \to x = 1 \Rightarrow x = y$

(I5) $(x \to y) \to y = (y \to x) \to x$

(L1) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$

(L2) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$

Definition 2.1.1

An algebra $l = (l; \to; ,0,1)$, ***I***= (I; →, ′,0,1) of type <2, 1,0, 0> is called an implicative algebra, if for every $x, y, z \in l$, x,y,z it satisfies the following conditions.

(1) $x \to (y \to z) = y \to (x \to z)$

(2) $1 \to x = x$

(3) $x \to 1 = 1$

(4) $x \to y = y' \to x'$

(5) $(x \to y) \to y = (y \to x) \to x$

(6) $0' = 1$

The binary operation clearly formalized by the implicative arrow "→" and every property of implicative algebra can be derived by using of the above conditions. Thus an implicative algebra is an equation defined algebra of type <2, 1, 0, 0>, where we designate the operation simply $x \to y$, for any $x,y \in I$ and we can read as "x arrow y". But other books can write $x \to y$ by xy and $x * y$.

Lemma 2.1.2

Every Boolean algebra (B; ∨,∧,',0,1) is an implicative algebra, that is for every $x,y \in B$ defined $x \to y = x' \vee y$.

Proof

Let **B** be a Boolean algebra and for every $x,y \in B$ define $x \to y = x' \vee y$.

Claim: B is an implicative algebra.

(1) $x\rightarrow(y\rightarrow z)=y\rightarrow(x\rightarrow z)$

Now, $x\rightarrow(y\rightarrow z)=x\rightarrow(y'\vee z)$

$=x'\vee(y'\vee z)$

$=(x'\vee y')\vee z$.........................(by associative of $\vee$)

$=(y'\vee x')\vee z$.........................(by commutative of $\vee$)

$=y'\vee(x'\vee z)$.........................(again by associative of $\vee$)

$=y\rightarrow(x\rightarrow z)$

(2) $1\rightarrow x=x$

Now, $1\rightarrow x=1'\vee x=0\vee x$

$=x$...................... since 0 is a least element of B

(3) $x\rightarrow 1=1$

Now, $x\rightarrow 1=x'\vee 1=1$......................since 1 is a greatest element of B

(4) $x\rightarrow y=y'\rightarrow x'$

Now, $x\rightarrow y=x'\vee y=y\vee x'=y'\rightarrow x'$

(5) $(x\rightarrow y)\rightarrow y=(y\rightarrow x)\rightarrow x$

Now, $(x\rightarrow y)\rightarrow y=(x'\vee y)\rightarrow y$

$=(x'\vee y)'\vee y$

$=(x\wedge y')\vee y$

$=y\vee(x\wedge y')$

$=(y\vee x)\wedge(y\vee y')$.......... (since (B, $\vee$,$\wedge$) is a distributive lattice)

$=(y\vee x)\wedge 1$

$=y\vee x$

Again, $(y\rightarrow x)\rightarrow x=(y'\vee x)\rightarrow x$

$=(y'\vee x)'\vee x$

$=(y\wedge x')\vee x$

$=x\vee(y\wedge x')$

$=(x\vee y)\wedge(x\vee x')$

$=(x \vee y) \wedge 1$

$=x \vee y$

But, $x \vee y = y \vee x$ and hence $(x \to y) \to y = (y \to x) \to x$.

Finally, in every Boolean algebra we have $1' = 0$ and $0' = 1$ and hence $(B; \vee, \wedge, \to, 0, 1, ')$ is an implicative algebra.

Definition 2.1.3

Let **I** be an implicative algebra then the relation $\leq$ defined on I by $x \leq y$ if and only if $x \to y = 1$, for all $x, y \in I$.

2.2 Properties of Implicative Algebras

Lemma 2.2.1

Let **I** be an implicative algebra, then for all $x, y \in I$, we have

(1) $x \to x = 1$

(2) $1' = 0$

Proof

Let **I** be an implicative algebra and $x, y \in I$

(1) $1 = (x \to 1) \to 1 = (1 \to x) \to x = x \to x$

(2) $1' = 1 \to 1'$

$= 0' \to 1'$

$= 1 \to 0$

$= 0$

Lemma 2.2.2

In an implicative algebra, for every $x, y, z \in I$ the following conditions hold.

(1) $0 \to x = 1$

(2) $x \to y = 1 = y \to x \Leftrightarrow x = y$

(3) $x \to y = 1$ and $y \to z = 1$, then $x \to z = 1$

(4) $x \leq y \Leftrightarrow z \to x \leq z \to y$ and $y \to z \leq x \to z$

(5) $((x \to y) \to y) \to y = x \to y$

(6) $(x \to y) \to [(y \to z) \to (x \to z)] = 1$

Proof

Let **I** be an implicative algebra and $x,y,z \in I$

(1) $0 \to x = x' \to 0'$

$= x' \to 1$

$=1$

(2) (⇒) Suppose $x \to y=1$ and $y \to x=1$

Now, $x=1 \to x = (y \to x) \to x$

$=(x \to y) \to y$

$=1 \to y=y$

Hence, $x \to y=1=y \to x$ imply $x=y$………………..(∗)

(⇐) suppose $x=y$

Now, $x \to y=x \to x$……………….…since $x=y$

$=1$

And, $y \to x=y \to y$………………….. since $x=y$

$=1$

Hence $x=y$ imply $x \to y=1=y \to x$……………....(∗∗)

Therefore, from (∗) and (∗∗) we have $x \to y=1=y \to x \Leftrightarrow x=y$.

(3) Let $x \to y=1$ and $y \to z=1$

Then $x \to z=x \to (1 \to z)$

$=x \to ((y \to z) \to z)$

$=x \to ((z \to y) \to y)$

$=(z \to y) \to (x \to y)$

$=(z \to y) \to 1$

$=1$

Hence if $x \to y=1$ and $y \to z=1$, then $x \to z=1$.

(4) (⇒) suppose $x \leq y$, then $x \to y=1$

Now, consider $(z \to x) \to (z \to y)=(x' \to z') \to (y' \to z')$

$=y' \to ((x' \to z') \to z')$

$=y'\to((z'\to x')\to x')$

$=(z'\to x')\to(y'\to x')$

$=(z'\to x')\to(x\to y)=1$

$=(z'\to x')\to 1=1$

Hence, $x\leq y$ imply $z\to x\leq z\to y$…………………………..(∗)

Similarly, consider $(y\to z)\to(x\to z)=x\to((y\to z)\to z)$

$=x\to((z\to y)\to y)$

$=(z\to y)\to(x\to y)$

$=(z\to y)\to 1=1$

Hence, $x\leq y$ imply $y\to z\leq x\to z$………………………..(∗∗)

(⇐) suppose $z\to x \leq z\to y$, then $(z\to x)\to(z\to y)=1$

Now, $x\to y=x\to(1\to y)=x\to((z\to x)\to(z\to y)\to y)$

$=(z\to x)\to(z\to y)\to(x\to y)$

$=(x'\to z')\to(y'\to z')\to(y'\to x')$

$=y'\to((x'\to z')\to z')\to(y'\to x')$

$=y'\to((z'\to x')\to x')\to(y'\to x')$

$=(z'\to x')\to(y'\to x')\to(y'\to x')$

$=(z'\to x')\to 1=1$

Hence, $z\to x \leq z\to y$ imply $x=y$……………………..(∗∗∗)

Consider $y\to z \leq x\to z$, then $(y\to z)\to(x\to z)=1$

Now, $x\to y = x\to(1\to y) =x\to((y\to z)\to(x\to z)\to y)$

$=(y\to z)\to(x\to z)\to(x\to y)$

$=x\to((y\to z)\to z)\to(x\to y)$

$=x\to((z\to y)\to y)\to(x\to y)$

$=(z\to y)\to(x\to y)\to(x\to y)$

$=(z\to y)\to 1=1$

Hence $y\to z \leq x\to z$ imply $x=y$…………………………(∗∗∗∗)

Therefore, from (∗), (∗∗). (∗∗∗) and (∗∗∗∗), we have $x\leq y \Leftrightarrow z\to x \leq z\to y$ and $y\to z \leq x\to z$.

(5) $((x \to y) \to y) \to y = (y \to (x \to y)) \to (x \to y)$

$= (x \to (y \to y)) \to (x \to y)$

$= (x \to 1) \to (x \to y)$

$= 1 \to (x \to y)$

$= x \to y$

Hence $((x \to y) \to y) \to y = x \to y$.

(6) $(x \to y) \to [(y \to z) \to (x \to z)] = (x \to y) \to [x \to ((y \to z) \to z)]$

$= (x \to y) \to [x \to ((z \to y) \to y)]$

$= (x \to y) \to [(z \to y) \to (x \to y)]$

$= (z \to y) \to [(x \to y) \to (x \to y)]$

$= (z \to y) \to 1 = 1$

Hence $(x \to y) \to [(y \to z) \to (x \to z)] = 1$.

Remarks 2.2.3: (1) From Lemma 2.2.1 (1), Lemma 2.2.2 (2) and (3), we have $x \to x = 1$, $x \to y = 1 = y \to x \Leftrightarrow x = y$ and if $x \to y = 1$ and $y \to z = 1$ then $x \to z = 1$ respectively. Therefore, it is clear that I is a partially ordered set.

(2) From definition 2.1.1 (3) and Lemma 2.2.2 (1), we have $x \to 1 = 1$ and $0 \to x = 1$ respectively, then $x \leq 1$ and $0 \leq x$. That is 1 and 0 are the greatest and the least elements of I respectively and hence I is bounded. Therefore, I is a bounded poset.

Lemma 2.2.4

Let **I** be an implicative algebra, then for $x \in I$, we have

(1) $(x')' = x$

(2) $x' = x \to 0$

Proof

Let **I** be an implicative algebra and $x \in I$

(1) $(x')' = 1 \to (x')'$

$= 0' \to (x')'$

$= x' \to 0$

$= x' \to 1'$

$= 1 \to x = x$

Hence, $(x')' = x$.

(2) $x'=1\to x'=(x')'\to 1'$

$=x\to 0$

Hence, $x'=x\to 0$

Theorem 2.2.6

In any implicative algebra **I**, the following hold for all $x,y\in I$.

$(x\vee y)'=x'\wedge y'$ and $(x\wedge y)'=x'\vee y'$.

Proof

Let **I** be an implicative algebra and $x,y\in I$

(1) Consider $(x\vee y)'\to x'\wedge y'=((x\to y)\to y)'\to((y'\to x')\to y'')'$

$=((x\to y)\to y)'\to((y'\to x')\to y)'$

$=((y'\to x')\to y)\to((x\to y)\to y)$

$=((x\to y)\to y)\to((x\to y)\to y)=1$

Then, $(x\vee y)'\leq x'\wedge y'$…………………………….$(*)$

And also ,$x'\wedge y'\to(x\vee y)'=((y'\to x')\to y'')'\to((x\to y)\to y)'$

$= ((x\to y)\to y)\to((y'\to x')\to y'')$

$= ((x\to y)\to y)\to((y'\to x')\to y)$

$= ((x\to y)\to y)\to((x\to y)\to y)=1$

Then, $x'\wedge y'\leq(x\vee y)'$…………………………$(**)$

Therefore, from $(*)$ and $(**)$,we have $(x\vee y)'=x'\wedge y'$.

(2) Consider $(x\wedge y)'\to x'\vee y'=(((y\to x)\to y')')'\to((x'\to y')\to y')$

$=((y\to x)\to y')\to((x'\to y')\to y')$

$=((y\to x)\to y')\to((y\to x)\to y')=1$

Hence, $(x\wedge y)'\leq x'\vee y'$…………………...…….$(*)$

And also, $x'\vee y'\to(x\wedge y)'=((x'\to y')\to y')\to(((y\to x)\to y')')'$

$=((x'\to y')\to y')\to((y\to x)\to y')$

$=((x'\to y')\to y')\to((x'\to y')\to y')=1$

Hence, $x'\vee y'\leq(x\wedge y)'$…………………………………$(**)$

Therefore, from $(*)$ and $(**)$, we have $(x\wedge y)'=x'\vee y'$.

Lemma 2.2.7

In any implicative algebra I, the following hold for all $x,y\in I$.

(a) $x\wedge y\leq x,y\leq x\vee y$

(b) $x\vee y$ is the least upper bound of $\{x,y\}$

(c) $x\wedge y$ is greatest lower bound of $\{x,y\}$

Proof

Let I be an implicative algebra and $x,y\in I$

(a) Consider $x\wedge y\to x=((y\to x)\to y')'\to x=x'\to((y\to x)\to y')$

$=(y\to x)\to(x'\to y')$

$=(y\to x)\to(y\to x)=1$

Hence, $x\wedge y\leq x$……………………….(∗)

Again, $x\wedge y\to y=((y\to x)\to y')'\to y=y'\to((y\to x)\to y')=(y\to x)\to(y'\to y')=1$

Hence, $x\wedge y\leq y$……………………...(∗∗)

Consider, $x\to x\vee y=x\to((x\to y)\to y)=(x\to y)\to(x\to y)=1$

Hence, $x\leq x\vee y$…………………..(∗∗∗)

$y\to x\vee y=y\to((x\to y)\to y)=(x\to y)\to(y\to y)=(x\to y)\to 1=1$

Hence, $y\leq x\vee y$…………………..(∗∗∗∗)

Therefore, from (∗), (∗∗), (∗∗∗) and (∗∗∗∗), we have $x\wedge y\leq x,y\leq x\vee y$.

(b) From (a), it can be observed that $x\vee y$ is an upper bound of $\{x,y\}$

Now, let u is an upper bound of x,y, then $x,y\leq u$ this implies $x\to u=1$ and $y\to u=1$

Now, $(x\vee y)\to u=((x\to y)\to y)\to u=((x\to y)\to y)\to(1\to u)$

$=((x\to y)\to y)\to((y\to u)\to u)$

$=((x\to y)\to y)\to((u\to y)\to y)$

$=(u\to y)\to(((x\to y)\to y)\to y)$

$=(u\to y)\to(x\to y)$…………by Lemma 2.2.2 (5)

$\geq x\to u=1$…………………....by Lemma 2.2.2 (6)

Hence , $(x\vee y)\to u=1\Rightarrow x\vee y\leq u$ .This shows that $x\vee y$ is the least upper bound of $\{x,y\}$.

(c) Clearly $x\wedge y$ is a lower bound of x,y. Now suppose that L is a lower bound of $\{x,y\}$

Then $L \leq x$ and $L \leq y$, which implies $L \to x = 1$ and $L \to y = 1$.

Consider $L \to (x \wedge y) = L \to ((y \to x) \to y')' = ((y \to x) \to y') \to L'$

$= ((y \to x) \to y') \to (1 \to L')$

$= ((y \to x) \to y') \to ((L \to y) \to L')$

$= ((y \to x) \to y') \to ((y' \to L') \to L')$

$= ((y \to x) \to y') \to ((L' \to y') \to y')$

$= (L' \to y') \to (((y \to x) \to y') \to y')$

$= (L' \to y') \to (((x' \to y') \to y') \to y')$

$= (L' \to y') \to (x' \to y')$

$\geq x' \to L' = L \to x = 1$

Thus $L \to (x \wedge y) = 1$, that is $L \leq (x \wedge y)$ and hence $x \wedge y$ is a greatest lower bound of $\{x, y\}$.

Corollary 2.2.10

In any implicative algebra I, the following conditions are valid for any

$x, y \in I$.

(1) $(x \vee y) \to z \leq x \to z$ and $(x \vee y) \to z \leq y \to z$

(2) $x \to z \leq (x \wedge y) \to z$ and $y \to z \leq (x \wedge y) \to z$

Proof

Let I be an implicative algebra and $x, y \in I$

(1) Consider $((x \vee y) \to z) \to (x \to z) = (((x \to y) \to y) \to z) \to (x \to z)$

$\geq x \to ((x \to y) \to y)$

$\geq (x \to y) \to (x \to y) = 1$

Hence , $((x \vee y) \to z) \to (x \to z) = 1$ it follows that $(x \vee y) \to z \leq x \to z$.

Again consider $((x \vee y) \to z) \to (y \to z) = (((x \to y) \to y) \to z) \to (y \to z)$

$\geq y \to ((x \to y) \to y)$

$\geq (x \to y) \to (y \to y)$

$\geq (x \to y) \to 1 = 1$

Hence, $((x \vee y) \to z) \to (y \to z) = 1$ it follows that $(x \vee y) \to z \leq y \to z$.

Therefore, $(x \vee y) \to z \leq x \to z$ and $(x \vee y) \to z \leq y \to z$.

(2) Consider $(x\to z)\to(x\wedge y)\to z=(x\to z)\to(((y\to x)\to y')'\to z)$

$=(x\to z)\to(z'\to((y\to x)\to y'))$

$=(x\to z)\to((y\to x)\to(z'\to y'))$

$=(x\to z)\to((y\to x)\to(y\to z))$

$=(y\to x)\to((x\to z)\to(y\to z))$

$=(y\to x)\to(y\to((x\to z)\to z))$

$=(y\to x)\to(y\to((z\to x)\to x))$

$=(y\to x)\to((z\to x)\to(y\to x))$

$=(z\to x)\to((y\to x)\to(y\to x))$

$=(z\to x)\to 1=1$

Hence, $(x\to z)\to(x\wedge y)\to z=1$ it follows that $x\to z\leq(x\wedge y)\to z$

Consider $(y\to z)\to((x\wedge y)\to z)=(y\to z)\to(((y\to x)\to y')'\to z)$

$=(y\to z)\to(z'\to((y\to x)\to y'))$

$=(y\to z)\to((y\to x)\to(z'\to y'))$

$=(y\to z)\to((y\to x)\to(y\to z))$

$=(y\to x)\to((y\to z)\to(y\to z))$

$=(y\to x)\to 1=1$

Hence, $(y\to z)\to((x\wedge y)\to z)=1$ it follows that $y\to z\leq(x\wedge y)\to z$.

Therefore, $x\to z\leq(x\wedge y)\to z$ and $y\to z\leq(x\wedge y)\to z$.

Theorem 2.2.11

In an implicative algebra I, the following conditions are satisfied. For any $x,y\in I$.

(a) $(x\vee y)\to z=(x\to z)\wedge(y\to z)$

(b) $(x\wedge y)\to z=(x\to z)\vee(y\to z)$

Proof

(a) From the above corollary 2.2.9 (1) and corollary 2.2.10 (1), we have

$(x\vee y)\to z\leq(x\to z)\wedge(y\to z)$…………...(∗)

And it remains to show that the other way $(x\to z)\wedge(y\to z)\leq(x\vee y)\to z$. Now

$[(x\to z)\wedge(y\to z)]\to(x\vee y)\to z$

$=[((x\to z)\to(y\to z))\to(x\to z)']'\to[((x\to y)\to y)\to z]$

$=[((x\to y)\to y)\to z]'\to[((x\to z)\to(y\to z))\to(x\to z)']$

$=[(x\to z)\to(y\to z)]\to[(((x\to y)\to y)\to z)'\to(x\to z)']$

$=[(x\to z)\to(y\to z)]\to[(x\to z)\to(((x\to y)\to y)\to z)]$

$=[y\to((x\to z)\to z)]\to[((x\to y)\to y)\to((x\to z)\to z)]$

$=((x\to y)\to y)\to[(y\to((x\to z)\to z)\to((x\to z)\to z))]$

$=((x\to y)\to y)\to[((((x\to z)\to z)\to y)\to y)]$

$=((x\to z)\to z)\to y\to[(((x\to y)\to y)\to y)]$

$=(((x\to z)\to z)\to y)\to(x\to y)$

$\geq x\to((x\to z)\to z)=(x\to z)\to(x\to z)=1$

Hence $(x\to z)\wedge(y\to z)\to(x\vee y)\to z=1$, then

Therefore, from $(*)$ and $(**)$ we have $(x\vee y)\to z=(x\to z)\wedge(y\to z)$.

(b) Clearly from the corollary 2.2.9 (2) and corollary 2.2.10 (2), we have

$(x\to z)\vee(y\to z)\leq(x\wedge y)\to z$……………………………….$(*)$

And it remains to show that the other way $(x\wedge y)\to z\leq(x\to z)\vee(y\to z)$. Now

$(x\wedge y)\to z\to(x\to z)\vee(y\to z)$

$=[((y\to x)\to y')'\to z]\to[((y\to z)\to(x\to z))\to(x\to z)]$

$=[z'\to((y\to x)\to y')]\to[((y\to z)\to(x\to z))\to(x\to z)]$

$=[(y\to x)\to(z'\to y')]\to[((y\to z)\to(x\to z))\to(x\to z)]$

$=[(y\to x)\to(y\to z)]\to[((y\to z)\to(x\to z))\to(x\to z)]$

$=(y\to z)\to(x\to z)\to[((y\to x)\to(y\to z))\to(x\to z)]$

$=x\to((y\to z)\to z)\to[((y\to x)\to(y\to z))\to(x\to z)]$

$=x\to((z\to y)\to y)\to[((y\to x)\to(y\to z))\to(x\to z)]$

$=(z\to y)\to(x\to y)\to[((y\to x)\to(y\to z))\to(x\to z)]$

$=(y\to x)\to(y\to z)\to[((z\to y)\to(x\to y))\to(x\to z)]$

$=(y\to x)\to(y\to z)\to[x\to((z\to y)\to y)\to(x\to z)]$

$=(y\to x)\to(y\to z)\to[x\to((y\to z)\to z)\to(x\to z)]$

$=(y\to x)\to(y\to z)\to[(y\to z)\to(x\to z)\to(x\to z)]$

$=(y \to x) \to (y \to z) \to [(y \to z) \to 1]$

$=(y \to x) \to (y \to z) \to 1 = 1$

Hence $(x \wedge y) \to z \to (x \to z) \vee (y \vee z) = 1$, then

$(x \wedge y) \to z \leq (x \to z) \vee (y \vee z)$…………..$(**)$

Therefore, from $(*)$ and $(**)$, we have $(x \wedge y) \to z = (x \to z) \vee (y \to z)$.

CONCLUSIONS

In this paper, based on the implication operator →, a partial order on non-empty I could be induced and generating boundary lattice on non-empty I with the greatest element and the least element is discussed. Then lattice implication algebra could be obtained if the operation "→" satisfies the defined axioms and I proved that implication algebra is lattice implication algebra. Moreover, we obtained the equivalent definition of lattice implication algebra of Xu with fewer axioms and proved that every implication algebra is a authometrized algebra of Swamy.

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